

Radiative Correction to the $\bar{\nu}_e$ (ν_e) Spectrum in β -Decay

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We derive an analytic expression for the $\mathcal{O}(\alpha)$ radiative correction to the $\bar{\nu}_e(\nu_e)$ spectrum in allowed β -decay. The $m \rightarrow 0$ limit is convergent and leads to a very simple result (m is the electron mass). This is in sharp contrast to the correction to the $e^-(e^+)$ spectrum, that diverges as $m \rightarrow 0$, an important difference that we explain on theoretical grounds. After discussing some of their general properties, we use the corrections to the $\bar{\nu}_e$ and e^- spectra to derive the corresponding correction to the $e^- \rightarrow \bar{\nu}_e$ conversion, a relation that is of considerable interest for reactor studies of neutrino oscillations.

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1 Introduction

The $\mathcal{O}(\alpha)$ radiative correction (R. C.) to the $e^-(e^+)$ spectrum has played an important role in precision studies of β -decay. It is described by a R. C. $(\alpha/2\pi)g(E, E_0)$, where E and E_0 are the energy and end-point energy of the $e^-(e^+)$ [1,2]. Here we have in mind the R. C. not contained in Fermi's Coulomb function $F(Z, E)$, which is separately included in the theory of β -decay. The effect of $(\alpha/2\pi)g(E, E_0)$ on the spectrum shape is significant, particularly for large values of E_0 [1,3]. It also gives important contributions to the theoretical calculations of the Ft-values of superallowed ($0^+ \rightarrow 0^+$) Fermi transitions [4] and the neutron lifetime [5]. We recall that the superallowed Fermi transitions play a crucial role in the precise verification of the unitarity of the Cabibbo-Kobayashi-Maskawa (CKM) matrix. In fact, the current analysis, which includes large short-distance effects [4], the R. C. $(\alpha/2\pi)g(E, E_0)$, and a number of more recent refinements [5,6], leads to the conclusion [7]:

$$|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 0.99995(61), \quad (1)$$

an impressive test of the three-generation Standard Theory at the level of its quantum corrections. It is also interesting to note that the overall electroweak corrections to Eq. (1) are of $\mathcal{O}(4\%)$, i. e. 66 times larger than the 0.061 % error.

The correction $(\alpha/2\pi)g$, in conjunction with the short-distance effects and refinements discussed in Refs. [4,5,6], has also been recently applied to the calculation of the muon capture rate in the singlet state of H [8].

The aim of the present paper is to derive the corresponding R. C. to the $\bar{\nu}_e(\nu_e)$ spectrum in allowed β -decay which, as explained below, is of considerable current interest.

An important difference with the R. C. to the $e^-(e^+)$ spectrum involves the inner bremsstrahlung (I. B.) contributions. In fact, the integration over the $e^-(e^+)$ momenta introduces a complicated dependence on the radiated photon energy that hinders the subsequent integration over the photon momenta. In Section 2 we develop and implement a convenient strategy to deal with this problem. The combination of the I. B. and virtual corrections leads then to the R. C. $(\alpha/2\pi)h(\hat{E}, E_0)$ to the $\bar{\nu}_e(\nu_e)$ spectrum, which we present in analytic form. Here $\hat{E} \equiv E_0 - K$, where K is the $\bar{\nu}_e(\nu_e)$ energy. We find that in the $m \rightarrow 0$ limit (m is the electron mass), $h(\hat{E}, E_0)$ converges and has a very simple form. This is in sharp contrast with $g(E, E_0)$, that diverges as $m \rightarrow 0$, an important difference that we explain on theoretical grounds.

In Section 3 we apply the R. C. $(\alpha/2\pi)g(E, E_0)$ and $(\alpha/2\pi)h(\hat{E}, E_0)$ to the conversion of the e^- spectrum to the $\bar{\nu}_e$ spectrum (" $e^- \rightarrow \bar{\nu}_e$ conversion" for short), a relation that is of considerable current interest for reactor studies of neutrino oscillations [9]. This leads to the analytic expression for the $\mathcal{O}(\alpha)$ R. C. to that important relation, which we illustrate with numerical examples.

2 Radiative Correction to the $\bar{\nu}_e(\nu_e)$ Spectrum

As in Ref. [1], we first consider the basic process of neutron β -decay, $n \rightarrow p + e^- + \bar{\nu}_e$, and neglect terms of $\mathcal{O}(\alpha E/M)$, where M is the nucleon mass.

The $\mathcal{O}(\alpha)$ virtual R. C., as well as the zeroth order transition probability, are obtained from those in the e^- spectrum by simply replacing $E \rightarrow \hat{E}, p \rightarrow \hat{p}, \beta \rightarrow \hat{\beta}$, where E, p , and m are the

e^- energy, momentum and mass, $\beta = p/E$, K is the $\bar{\nu}_e$ energy and \hat{E} , \hat{p} , and $\hat{\beta}$ are defined by

$$\hat{E} \equiv E_0 - K; \quad \hat{p} \equiv \left(\hat{E}^2 - m^2 \right)^{1/2}; \quad \hat{\beta} \equiv \hat{p}/\hat{E}. \quad (2)$$

The reason for this result is that, in deriving the $\bar{\nu}_e$ spectrum, one integrates over the e^- phase space factor $d^3p/2E$ rather than the $\bar{\nu}_e$'s $d^3K/2K$. In these two contributions the conservation of energy is expressed by $\delta(E_0 - K - E)$ and the phase space integrations lead to the relation explained above.

We now turn our attention to the inner bremsstrahlung (I. B.) contributions, which present more subtle and theoretically interesting differences with the e^- spectrum.

Calling ε_γ the radiated photon energy, the I. B. amplitude involves two terms: the first one behaves as $1/\varepsilon_\gamma$ as $\varepsilon_\gamma \rightarrow 0$; the second one is of $\mathcal{O}(\varepsilon_\gamma^0)$. The square of the absolute value of the first term, combined with the $d^3k/2\varepsilon_\gamma$ photon phase space factor (k is the photon momentum) leads to the infrared (I. R.) divergent contribution. The interference of the two terms and the square of the absolute value of the second one are I. R. convergent and are evaluated separately.

As in [1], we regularize the I. R. divergence by introducing an infinitesimal photon mass λ_{\min} , so that $\varepsilon_\gamma = (k^2 + \lambda_{\min}^2)^{1/2}$. After integrating over the proton and electron momenta, as well as the $\bar{\nu}_e$ direction, the I. R. contribution is proportional to $\int_0^{k_m} (d^3k/2\varepsilon_\gamma^3) f(\varepsilon_\gamma, v)$, where

$$f(\varepsilon_\gamma, v) = pE \left\{ (1/\beta v) \ln[(1 + \beta v)/(1 - \beta v)] - (1 - \beta^2)/(1 - \beta^2 v^2) - 1 \right\}, \quad (3)$$

$v = k/\varepsilon_\gamma$, $k_m = \hat{E} - m$ is the maximum photon energy for given K , and it is understood that

$$E = \hat{E} - \varepsilon_\gamma, \quad (4)$$

which follows from energy conservation.

The ε_γ variable in $f(\varepsilon_\gamma, v)$ denotes the ε_γ dependence contained in E via Eq. (4) and, therefore, also in p , β , and Eq. (3). The complicated dependence of $f(\varepsilon_\gamma, v)$ on ε_γ and v hinders the subsequent integration over the photon momenta. To deal with this problem, a convenient strategy is to split $f(\varepsilon_\gamma, v)$ according to

$$f(\varepsilon_\gamma, v) = f(0, v) + [f(\varepsilon_\gamma, v) - f(0, v)]. \quad (5)$$

The I. R. divergence arises from $f(0, v)$, a function that only involves the photon momenta through its dependence on v . Changing the integration variable from k to v , the integral $\int_0^{k_m} (d^3k/2\varepsilon_\gamma^3) f(0, v)$ can be performed analytically and leads to a function of $\hat{\beta}$ and $\ln(k_m/\lambda_{\min})$. Its combination with the virtual $\mathcal{O}(\alpha)$ R. C. is well defined since, as expected, the I. R. divergences cancel exactly.

The integration of the second term in Eq. (5) is I. R. convergent, so we set $\lambda_{\min} = 0$ ($v = 1$), and replace $f(\varepsilon_\gamma, v) - f(0, v) \rightarrow f(k, 1) - f(0, 1)$. In this latter expression, the $\varepsilon_\gamma = k$ dependence is contained in E and, therefore, also in p , β , and $f(k, 1)$ (cf. Eqs. (3, 4)).

Changing the variable of integration from k to E via Eq. (4), the integral $\int d^3k/2k^3 [f(k, 1) - f(0, 1)]$ can be also performed analytically. In particular, we note that the result of this integration includes the integral

$$I = \int_{m/\hat{E}}^1 dt \ln(1 - t)/(t^2 - m^2/\hat{E}^2)^{1/2}, \quad (6)$$

for which we have derived the compact analytic representation

$$I = -\text{Li}_2\left(2\hat{\beta}/(1+\hat{\beta})\right) + \tanh^{-1}\hat{\beta} \ln\left(2\hat{\beta}^2/(1+\hat{\beta})\right), \quad (7)$$

where

$$\text{Li}_2(z) = -\int_0^z (dt/t) \ln(1-t) \quad (8)$$

is the dilogarithm.

As mentioned before, the d^3k integration of the two remaining contributions, namely the interference of the two terms in the I. B. amplitude and the absolute value square of the second, are I.R. convergent. Changing again the variable of integration from k to E via Eq. (4), they can also be performed analytically.

Combining all the $\mathcal{O}(\alpha)$ contributions discussed above, we obtain the radiatively corrected expression for the $\bar{\nu}_e$ spectrum:

$$dP_\nu = dP_\nu^0 \left[1 + (\alpha/2\pi)h(\hat{E}, E_0) \right], \quad (9)$$

where

$$dP_\nu^0 = A \hat{p} \hat{E} F(Z, \hat{E}) K^2 dK \quad (10)$$

is the zeroth order spectrum,

$$\begin{aligned} h(\hat{E}, E_0) = & 3 \ln\left(\frac{m_p}{m}\right) + \frac{23}{4} - \frac{8}{\hat{\beta}} \text{Li}_2\left(\frac{2\hat{\beta}}{1+\hat{\beta}}\right) + 8 \left(\frac{\tanh^{-1}\hat{\beta}}{\hat{\beta}} - 1\right) \ln\left(\frac{2\hat{E}\hat{\beta}}{m}\right) \\ & + 4 \frac{\tanh^{-1}\hat{\beta}}{\hat{\beta}} \left[\frac{7+3\hat{\beta}^2}{8} - 2 \tanh^{-1}\hat{\beta} \right], \end{aligned} \quad (11)$$

m_p is the proton mass, and $\hat{E}, \hat{p}, \hat{\beta}, \text{Li}_2(z)$ are defined in Eqs. (2,8). In Eq. (10), $F(Z, \hat{E})$ is the Fermi Coulomb function evaluated at $E = \hat{E}$ and A is a constant independent of K . In neutron β -decay, for example, $A = (g_V^2 + 3g_A^2)/2\pi^3$, where $g_V = G_F V_{ud}$ and g_A are the vector and axial vector coupling constants. In analogy with the e^- spectrum case [1], Eqs. (9-11) can be applied to allowed nuclear β decay using an independent particle model of the nucleus. In that case $g_V^2 + 3g_A^2$ is replaced by $g_V^2 |M_F|^2 + g_A^2 |M_{GT}|^2$, where M_F and M_{GT} are the Fermi and Gamow-Teller matrix elements. In analogy with $g(E, E_0)$, Eq. (11) is valid for both $\bar{\nu}_e$ and ν_e transitions.

Aside from the corrections $(\alpha/2\pi)g(E, E_0)$ and $(\alpha/2\pi)h(\hat{E}, E_0)$, in the Standard Theory there are large additional electroweak corrections which give very important contributions to the Ft-values of the superallowed Fermi transitions and the neutron lifetime. However, they are constants independent of K and E , so they don't affect the shape of the $\bar{\nu}_e$ and e^- spectra. For this reason, they are not included in the present discussion.

Eq. (11) simplifies considerably in the $m \rightarrow 0$ ($\hat{\beta} \rightarrow 1$) limit. Noting that $\tanh^{-1}\hat{\beta} = \ln\left[\hat{E}(1+\hat{\beta})/m\right]$ and $\text{Li}_2(1) = \pi^2/6$, we readily find:

$$\lim_{m \rightarrow 0} h(\hat{E}, E_0) = 3 \ln\left(\frac{m_p}{2E_0}\right) + \frac{23}{4} - \frac{4\pi^2}{3} - 3 \ln(1-y), \quad (12)$$

where $y = K/E_0$. Thus, the $m \rightarrow 0$ limit of $h(\hat{E}, E_0)$ is convergent and has a very simple form. This is in sharp contrast with the behavior of $g(E, E_0)$, the function that describes the R. C. to the $e^-(e^+)$ spectrum. In fact, in the relativistic $m^2 \ll E^2$ limit

$$\begin{aligned}
g(E, E_0) \rightarrow & 3 \ln \left(\frac{m_p}{2E_0} \right) - \frac{3}{4} - \frac{2\pi^2}{3} \\
& + 4 (\ln x - 1) \left[\frac{1-x}{3x} - \frac{3}{2} + \ln \left(\frac{1-x}{x} \right) \right] + \frac{(1-x)^2}{6x^2} \ln x \\
& + \ln \left(\frac{2E_0}{m} \right) \left[\frac{4(1-x)}{3x} - 3 + \frac{(1-x)^2}{6x^2} + 4 \ln \left(\frac{1-x}{x} \right) \right], \quad (13)
\end{aligned}$$

where $x = E/E_0$. Eq. (13) can be gleaned from the expression between curly brackets in Eq. (4.4) of Ref. [3], after subtracting $6 \ln(\lambda/m_p) + 9/4$ (cf. footnote 23 in Ref. [1]). Thus, the correction to the electron spectrum diverges logarithmically in the $m \rightarrow 0$ limit! On the other hand, if one multiplies Eq. (13) by $x^2(1-x)^2$, which describes the energy dependence of the zeroth order probability in the $m \rightarrow 0$ limit, and integrates over $0 \leq x \leq 1$, the cofactor of $\ln(2E_0/m)$ vanishes, so that the correction to the inverse lifetime is indeed free from the electron ‘‘mass singularity’’ [3,10,11].

The fact that $h(\hat{E}, E_0)$ is convergent in the $m \rightarrow 0$ limit while $g(E, E_0)$ is not, can be explained by the following theoretical argument. For given K , in the $m \rightarrow 0$ limit all collinear $e - \gamma$ configurations become energy degenerate with energy $E_0 - K$ and generally give rise to mass singularities (cf. Eq. (3)). An elementary, but powerful theorem in Quantum Mechanics leads to the conclusion that these singularities are cancelled in the power series expansions of transition probabilities if the latter are summed over an appropriate ensemble of such degenerate states [11]. In deriving the corrections to the $\bar{\nu}_e(\nu_e)$ spectrum, we perform the d^3p and d^3k integrations, so indeed we sum over the set of collinear $e - \gamma$ configurations that become energy degenerate in the $m \rightarrow 0$ limit. Thus, according to this theorem, $h(\hat{E}, E_0)$ should be free of $\ln m$ singularities, as we find. In contrast, this is not the case in the derivation of the electron spectrum, since the d^3p integration is not carried out. As a consequence, $g(E, E_0)$ (Eq. (13)) explicitly exhibits a $\ln m$ singularity. Analogous examples of mass singularities in the differential spectrum, and their cancellation in the lifetime, integrated asymmetry and some partial decay rates in muon decay are extensively discussed in Ref. [3].

A useful consistency check of Eqs. (12,13) is to multiply Eq. (12) by $y(1-y)$, Eq. (13) by $x(1-x)$, and integrate over x and y from 0 to 1. The two results should be equal since both lead to the correction to the inverse lifetime, a fact that we have verified to high accuracy. Eq. (12) can also be derived directly by neglecting the terms in the integrands that vanish in the $m \rightarrow 0$ limit. In that approximation the phase space integrations are very simple, and we find that the $\ln m$ singularities from the various contributions indeed cancel, leading again to Eq. (12).

As an illustrative example, we consider the case $E_0 = 8$ MeV. Using Eq. (11), we find that the R. C. $(\alpha/2\pi)h(\hat{E}, E_0)$ to the $\bar{\nu}_e(\nu_e)$ spectrum increases the transition probability by 1.17 % at $y = 0.8$ ($K = 6.4$ MeV) and by 0.64 % at $y = 0.2$ ($K = 1.6$ MeV). This behavior is reflected in the simpler expression in Eq. (12), which shows a positive, monotonically increasing function of y . Using the exact expression for $g(E, E_0)$ (Eq. (20b) of Ref. [1]), we find that the

R. C. $(\alpha/2\pi)g(E, E_0)$ to the $e^-(e^+)$ spectrum decreases the transition probability by 1.13 % at $x = 0.8$ ($E = 6.4$ MeV) and increases it by 2.90 % at $x = 0.2$ ($E = 1.6$ MeV). This behavior can be qualitatively understood by noting that the cofactor of $\ln(2E_0/m)$ in Eq. (13) is negative for $x \gtrsim 0.44$ and positive for $x \lesssim 0.44$. As a consequence, the R. C. decreases (increases) the transition probability for high (low) energy e^- or e^+ , an effect that becomes sharper as E_0 increases. A similar behavior has been found in muon decay [3].

In many cases, Eqs. (12,13) provide very good and useful approximations to the exact expressions for $h(\hat{E}, E_0)$ and $g(E, E_0)$. Their domains of validity are $(E_0 - K)^2 \gg m^2$ and $E^2 \gg m^2$, respectively. As shown in Ref. [1], the $\mathcal{O}(\alpha)$ R. C. to the shape of the $e^-(e^+)$ spectrum is not affected by strong interactions if terms of $\mathcal{O}(\alpha E/M)$ are neglected. The same conclusion applies to the $\mathcal{O}(\alpha)$ R. C. to the $\bar{\nu}_e(\nu_e)$ spectrum.

3 Radiative Correction to the $e^- \rightarrow \bar{\nu}_e$ Conversion

Including the $\mathcal{O}(\alpha)$ R. C., the theoretical expressions for the e^- and $\bar{\nu}_e$ spectra in allowed β -decay can be cast in the form

$$\frac{dP_e}{dE} = f_e(E, E_0), \quad (14)$$

$$\frac{dP_\nu}{dK} = f_\nu(K, E_0), \quad (15)$$

where

$$f_e(E, E_0) = A p E (E_0 - E)^2 F(Z, E) \left[1 + \frac{\alpha}{2\pi} g(E, E_0) \right], \quad (16)$$

$$f_\nu(K, E_0) = A \hat{p} \hat{E} K^2 F(Z, \hat{E}) \left[1 + \frac{\alpha}{2\pi} h(\hat{E}, E_0) \right], \quad (17)$$

and \hat{E} and \hat{p} are defined in Eq. (2). Comparing Eqs. (16,17) and recalling that $\hat{E} = E_0 - K$, we see that

$$f_\nu(K, E_0) = f_e(\hat{E}, E_0) \left[1 + \frac{\alpha}{2\pi} h(\hat{E}, E_0) \right] / \left[1 + \frac{\alpha}{2\pi} g(\hat{E}, E_0) \right], \quad (18)$$

or, neglecting terms of order $\mathcal{O}(\alpha^2)$:

$$f_\nu(K, E_0) = f_e(\hat{E}, E_0) \left[1 + \frac{\alpha}{2\pi} \left(h(\hat{E}, E_0) - g(\hat{E}, E_0) \right) \right]. \quad (19)$$

Using Eq. (11) and the exact expression for $g(E, E_0)$ (Eq. (20b) of Ref. [1]), we find

$$\begin{aligned} h(\hat{E}, E_0) - g(\hat{E}, E_0) &= \frac{13}{2} - \frac{4}{\hat{\beta}} \text{Li}_2 \left(\frac{2\hat{\beta}}{1+\hat{\beta}} \right) \\ &+ 4 \left(\frac{\tanh^{-1}\hat{\beta}}{\hat{\beta}} - 1 \right) \left[\frac{11}{6} + \ln \left(\frac{2E_0}{m} \right) + 2 \ln \hat{\beta} + 2 \ln(1-y) - \ln y - \frac{1}{3(1-y)} \right] \\ &+ \frac{\tanh^{-1}\hat{\beta}}{\hat{\beta}} \left[\frac{3-\hat{\beta}^2}{2} - 4 \tanh^{-1}\hat{\beta} - \frac{y^2}{6(1-y)^2} \right], \end{aligned} \quad (20)$$

where $\hat{\beta}$ is defined in Eq. (2) and we recall that $y \equiv K/E_0$.

Eqs. (19,20) describe the conversion of the e^- spectrum in a specific decay into the corresponding $\bar{\nu}_e$ spectrum ($e^- \rightarrow \bar{\nu}_e$ conversion for short) when $\mathcal{O}(\alpha)$ R. C. are included.

As an illustrative example, we consider again the case $E_0 = 8$ MeV. Using Eqs. (19,20) we find that, for a given e spectrum $f_e(E, E_0)$, the R. C. $(\alpha/2\pi) [h(\hat{E}, E_0) - g(\hat{E}, E_0)]$ decreases $f_\nu(K, E_0)$ by 1.74 % at $y = 0.8$ ($K = 6.4$ MeV) and increases it by 1.77 % at $y = 0.2$ ($K = 1.6$ MeV). Thus, the correction factor $[1 + (\alpha/2\pi)(h(\hat{E}, E_0) - g(\hat{E}, E_0))]$ in Eq. (19) is 3.57 % larger at $K = 1.6$ MeV than at $K = 6.4$ MeV. This behavior reflects the dominant contribution of $-g(\hat{E}, E_0)$. As a consequence, the R. C. decreases (increases) $f_\nu(K, E_0)$ for high (low) K , an effect that becomes sharper as E_0 increases.

4 Conclusions

In summary, we have derived an analytic expression for the $\mathcal{O}(\alpha)$ R. C. to the $\bar{\nu}_e(\nu_e)$ spectrum in allowed β -decay (Eqs. (9-11)). We have found that the $m \rightarrow 0$ limit of this correction is convergent and leads to a very simple expression (Eq. (12)). This is in sharp contrast to the $\mathcal{O}(\alpha)$ R. C. to the $e^-(e^+)$ spectrum, that diverges as $m \rightarrow 0$, an important difference that we have explained on theoretical grounds. We have then used our results to derive the $\mathcal{O}(\alpha)$ R. C. to the $e^- \rightarrow \bar{\nu}_e$ conversion (Eqs. (19,20)), a relation that plays an important rôle in reactor studies of neutrino oscillations.

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